Int. J. Solids Structures Vol. 19, No. 1, pp. 33–54, 1983
Printed in Great Britain.
0020-7683/83/010033-22$03.00/0

A NEW APPROACH (INCLUDING SHEAR LAG) TO ELEMENTARY MECHANICS OF MATERIALS

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(*Received* 29 *December* 1981)

Abstract—It is shown that the assumption of an elastic, transversely rigid, material gives extended solutions of the Saint-Venant flexure and torsion problems, rigorously applicably to a linearly varying shear force or torsional couple, respectively.

The theory is applied in both cases to practical examples. It enables to establish naturally the theory of the "Timoshenko beam" and shows the position held in Mechanics of Materials by the approximate Timoshenko–Vlasov warping torsion theory. In summary, the new approach gives a more scientific foundation to the results of Mechanics of Materials. The results obtained are especially interesting for materials weak in shear, i.e. those for which the ratio of the elastic moduli, E/G, is large.

1. INTRODUCTION

Since 1821, Theory of Elasticity and Mechanics of Materials have developed in parallel. The second doctrine is based on the neglect of the transverse direct stresses, the assumption that the cross sections are rigid in their plane, which requires the presence of infinitely many cross frames, absolutely rigid in their own plane and perfectly deformable out of this plane. Under these conditions, Hooke's law reduces to the two extremely simple relations $\sigma = E\epsilon$, $\tau = G\gamma$. In addition, elementary Mechanics of Materials neglects usually the effect of shear deformations on the deformation of beams bent by transverse forces, as well as the shear lag, that is the unequal distribution of the direct stresses in thin flanges composing the walls of plate–and box girders.

(1) It can be shown that foregoing results are in reality rigorous results of Theory of Elasticity for an elastic orthotropic material, that is transversely rigid.

(2) It can also be shown that Navier's formula $\sigma = My/I$, which is known to be valid for a linearly varying bending moment, is still valid for uniformly distributed forces producing a quadratic bending moment, provided it is corrected by a function $\Delta\sigma(x, y)$, identical in all cross sections, and that will be called "distributed shear lag".

(3) The new approach gives a means to determine this "distributed shear lag", which differs from the usual shear lag by the fact that it does not tend towards zero at the two ends of the beam. It can therefore be expected to be a good approximation of the actual shear lag, provided the beam is sufficiently elongated.

(4) The new approach enables to establish that Timoshenko's equation for the deflection of the axis of a beam loaded by transverse forces

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} + \frac{1}{GA'}\frac{d^2M}{dx^2}$$

is rigorous in the case of uniformly distributed transverse forces.

(5) In the case of torsion of a prismatic bar by a linearly varying torque, the new approach shows that Saint-Venant theory of torsion is still applicable, provided a certain distribution of axial direct stresses (the same in all sections) is introduced. If the torque M_Z varies arbitrarily along the axis, these axial direct stresses vary also with z and are accompanied by corrective shear stresses $\Delta\tau_{xy}$, $\Delta\tau_{yz}$, which, in the case of a thin-walled beam with open cross section, become identical to the σ and τ stresses of the Timoshenko–Vlasov warping torsion theory.

(6) Finally, an important advantage of the new approach is that it is valid irrespectively of the value of the ratio $k = E/G$ of the elastic moduli. The results are therefore applicable to materials weak in shear and, in particular, to shear-soft tubes representing approximately multi-story buildings in the form on framed tubes.

2. THE TECHNICAL THEORY OF BEAMS, CONSIDERED AS A RIGOROUS ELASTIC THEORY FOR AN ADEQUATE ORTHOTROPIC MATERIAL

The basic assumptions of the theory of beams developed in Elementary Mechanics of Materials are

(1) transverse direct stresses σ_x and τ_{xy} are neglected;

(2) the corresponding strains ϵ_x, ϵ_y and γ_{xy} are zero, so that the cross sections may be considered as rigid in their planes;

(3) in thin-walled beams, this rigidity postulates the existence of infinitely many transverse frames, which should be rigid in their plane but could freely distort out of this plane;

(4) The bending stresses are approximately given by Navier's formula

$$\sigma = \frac{M_x y}{I_x}$$

and the shear stresses τ are derived from the σ by pure equilibrium considerations. Various approximate theories have been proposed. They all are based on the formula (Fig. 1)

$$\mathrm{d}R = \frac{TS\,\mathrm{d}x}{I}$$

giving the longitudinal shear force acting on the horizontal section *abcd* (Fig. 1).

It can easily be shown that Mechanics of Materials can be established on a rigorous basis by considering it as Theory of Elasticity applied to an orthotropic, transversely rigid material.

This approach has following advantages:

(1) bring some new results that have already been discussed in the introduction;

(2) clarify the relations between the technical doctrine called Mechanics of Materials and Theory of Elasticity;

(3) extend to Mechanics of Materials the rigorous variational theorems of Theory of Elasticity as well as Kirchhoff's unicity theorem.

In his book ([1] pp. 20, 21), Leknitzkii shows that, for an orthogonally anisotropic material, the elastic stress–strain relations are

$$\left\{\begin{aligned} \epsilon_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z \\ \epsilon_y &= -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_3}\sigma_z \\ \epsilon_z &= -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z \end{aligned}\right. \tag{2.1}$$

$$\gamma_{yz} = \frac{1}{G_{23}}\tau_{yz};\ \gamma_{xz} = \frac{1}{G_{13}}\tau_{xz};\ \gamma_{xy} = \frac{1}{G_{12}}\tau_{xy}. \tag{2.2}$$

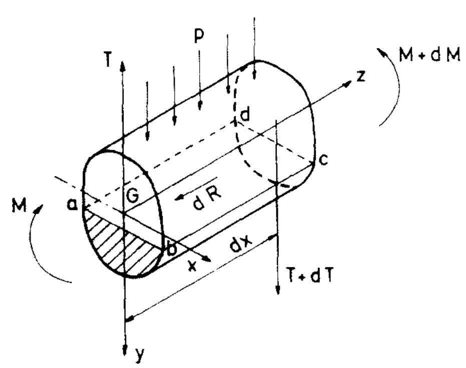

Fig. 1. Transversely loaded bar.

The 12 elastic constants E_1, E_2, E_3, G_1, G_2, G_3, ν_{12}, ν_{21}, ν_{13}, ν_{31}, ν_{23}, ν_{32} are connected by the three reciprocity relations

$$E_1\nu_{21} = E_2\nu_{12};\ E_2\nu_{32} = E_3\nu_{23};\ E_3\nu_{13} = E_1\nu_{31}, \tag{2.3}$$

which leaves 9 independent constants.

To carry out the classical assumption of Mechanics of Materials, namely the indeformability of the cross section, we must choose the elastic constants in such a way as to suppress all deformations in the plane of the cross sections; this requires

$$\epsilon_x = 0;\ \epsilon_y = 0;\ \gamma_{xy} = 0 \tag{2.4}$$

and consequently,

$$E_2 = E_3 = \infty,\ \nu_{12} = \nu_{21} = \nu_{13} = \nu_{31} = 0;\ G_{12} = 0. \tag{2.5}$$

On the other hand, we wish to keep the beam isotropic under transverse shear. The corresponding shear moduli must therefore be equal

$$G_{13} = G_{23} = G. \tag{2.6}$$

In these conditions, eqns (2.1) and (2.2) reduce to

$$\begin{cases} \epsilon_x = \epsilon_y = \gamma_{xy} = 0 \\ \epsilon_z = \dfrac{\sigma_z}{E} \\ \gamma_{xz} = \dfrac{\tau_{xz}}{G} \\ \gamma_{yz} = \dfrac{\tau_{yz}}{G}. \end{cases} \tag{2.7}$$

Moduli E and G are now entirely independent, which enables to consider materials very deformable in shear, in which we shall see that the distributed shear lag will be especially large.

The counterpart, in Theory of Elasticity, of the elementary theory of beams, is the general so-called "Saint-Venant problem", which amounts to find the stress, strain and displacement fields in prismatic bodies acted upon (Fig. 2) by forces and moments applied to their end sections. More precisely, the prismatic beam of length 1 is subjected at its free end to:

(1) a normal force N

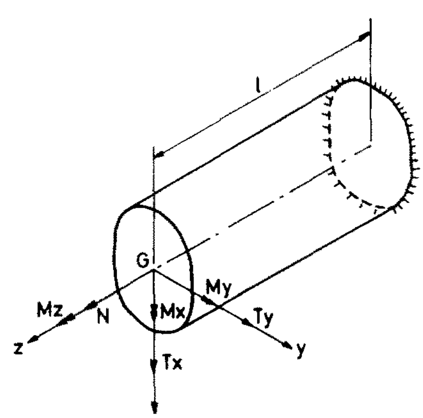

Fig. 2. Internal resultants in a bar.

(2) a vertical transverse force P_x producing a vertical shear force $T_x = P_x$ and a linearly varying moment $M_y = T_x z$.

(3) an horizontal transverse force P_y producing an horizontal shear force $T_y = P_y$ and a linearly varying moment $M_x = P_y z$.

(4) a constant torsional couple of moment M_z.

It will be shown in what follows that, when the material is transversely rigid as specified hereabove, it is possible to find the rigorous solution, not only for above loading, but also for an eccentric vertical loading uniformly distributed with the intensity p along the z axis. Led d be the distance of these loads to the shear centre (Fig. 3). Then, obviously, the most general loading combining the loadings of Fig. 2 and 3 is

$$N(z) = N \tag{2.8}$$

$$T_x(z) = T_0 - pz$$

$$M_y(z) = -\frac{pz^2}{2} + T_0 z + C \tag{2.9}$$

and similar terms in T_y, M_x for an eccentric horizontal loading

$$M_z(z) = M_{z0} + pdz. \tag{2.10}$$

The axial loading case has the trivial solution:

$$\sigma_x = \frac{N}{A}, \ \epsilon_x = \frac{N}{EA}.$$

The two cases of uniform transverse loading are basically the same and we shall only study, in Section 6, the first one, defined by the internal resultants

$$T_x(z) = T_0 - pz; \ M_y(z) = -\frac{pz^2}{2} + T_0 z + C \tag{2.9}$$

Finally, the torsional problem under a linearly varying torsional couple

$$M_z(z) = M_{z0} + zM'_z \text{ (with } M'_z = pd) \tag{2.10}$$

deserves a special study (Section 3).

The two celebrated Saint-Venant problems correspond to $p = 0$; the first one is the uniform torsion of a prism under a constant torsional couple. The second is the bending with shear under a constant shear force $T_0 = P_x$ and a linearly varying moment $M_y = T_0 z$.

We shall see that in both problems above, the solution is similar to that obtained by Saint-Venant, which will be used as leading thread.

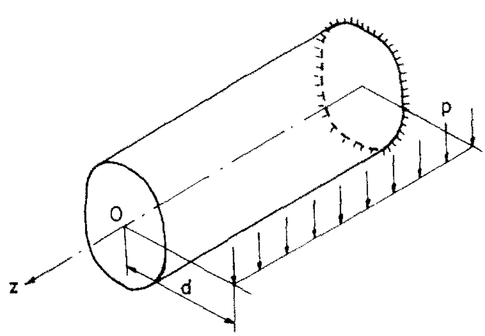

Fig. 3. Bent and twisted bar.

In both cases, the new solutions involve constant transverse volume forces. These forces are obviously necessary to introduce the transverse distributed force of constant intensity p. Whether this force is produced by constant transverse surface tractions T_x, T_y, or by constant transverse volume forces is completely irrelevant, because, if the transverse stresses σ_x, σ_y are obviously different in both cases, the deformations are not affected, because the material is transversely rigid. Besides, in Mechanics of Materials, no attention is paid to these transverse stresses.

3. TORSION OF AN ORTHOTROPIC PRISMATIC BAR, TRANSVERSELY RIGID, UNDER THE ACTION OF A LINEARLY VARYING TORQUE

3.1 *The displacement, strain and stress fields*

We choose as origin the shear center 0 of the built-in section Oxy (Fig. 4). We must obviously keep Saint-Venant's hypothesis of a rotation "en bloc" of the cross sections [2], because here these cross sections are rigid. We add to the corresponding displacements u, v, where β is the current angle of torsion, a warping displacement w proportional to the unit torsion $\theta = \mathrm{d}\beta/\mathrm{d}z$. We have thus

$$\begin{cases} u = -\beta y \\ v = +\beta x \\ w = \theta\psi(x, y). \end{cases} \tag{3.1}$$

With

$$M_z = M_{z0} + zM'_z \tag{3.2}$$

$$\theta = \theta_0 + \theta' z = \frac{M_z}{C}, \tag{3.3}$$

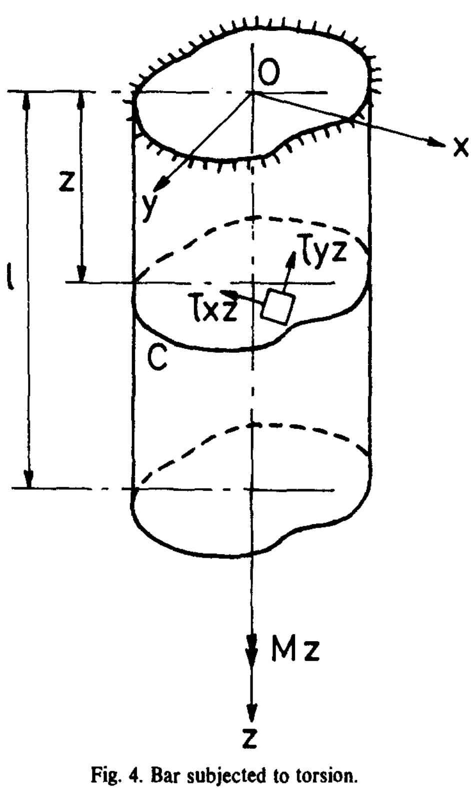

Fig. 4. Bar subjected to torsion.

where C is the torsional rigidity, and

$$\beta = \int_0^z \theta \mathrm{d}z = \theta' \frac{z^2}{2} + \theta_0 z, \tag{3.4}$$

the displacement field may be written

$$\begin{cases} u = -\left(\theta' \dfrac{z^2}{2} + \theta_0 z\right) y \\ v = \left(\theta' \dfrac{z^2}{2} + \theta_0 z\right) x \\ w = (\theta_0 + \theta'_z)\psi(x, y). \end{cases} \tag{3.5}$$

The following components of the strain tensor

$$\epsilon_x = \frac{\partial u}{\partial x} = 0;\ \epsilon_y = \frac{\partial v}{\partial y} = 0;\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{3.6}$$

as in the Saint-Venant problem.

$$\epsilon_z = \frac{\partial w}{\partial z} = \theta' \psi(x, y) \text{ whence } \sigma_z = E\epsilon_z = E\theta' \psi(x, y). \tag{3.7}$$

The solution involves thus a distribution of axial direct stresses which is the same in all sections, and which can be considered as the distributed shear lag of present problem. Finally:

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = (\theta' z + \theta_0)\left(y + \frac{\partial \psi}{\partial x}\right);\ \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = (\theta' z + \theta_0)\left(-x + \frac{\partial \psi}{\partial y}\right) \tag{3.8}$$

whence

$$\begin{aligned} \tau_{xz} &= G(\theta' z + \theta_0)\left(-y + \frac{\partial \psi}{\partial x}\right) \\ \tau_{yz} &= G(\theta' z + \theta_0)\left(x + \frac{\partial \psi}{\partial y}\right). \end{aligned} \tag{3.9}$$

The equations of compatibility are identically satisfied because the solution starts from the continuous displacement field (3.1). We must therefore consider only the equations of internal equilibrium, and thereafter the statical boundary conditions.

3.2 *Equations of equilibrium*

Suppose that there are body forces $\underline{F}$, whose z component is zero. Then, the equations of equilibrium reduce to

$$\frac{\partial \tau_{xz}}{\partial z} + F_x = 0;\ \frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \tag{3.10}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{3.11}$$

Equation (3.11) will be satisfied if we introduce Prandtl's stress function $\phi(x, y)$ as follows:

$$\tau_{xz} = G\theta \frac{\partial \phi(x, y)}{\partial y};\ \tau_{yz} = -G\theta \frac{\partial \phi(x, y)}{\partial x} \tag{3.12}$$

Equating (3.9) and (3.12), we have:

$$G\theta\left(-y+\frac{\partial\psi}{\partial x}\right)=G\theta\frac{\partial\phi}{\partial y} \tag{3.13}$$

$$G\theta\left(x+\frac{\partial\psi}{\partial y}\right)=-G\theta\frac{\partial\phi}{\partial x}. \tag{3.14}$$

We eliminate ψ by differentiating (3.13) with respect to y, (3.14) with respect to x, and subtracting; this gives

$$\boxed{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-2} \tag{3.15}$$

The boundary condition reduces here, like in Saint-Venant problem (if, for simplicity, we suppose the cross section to be simply connected) to

$$\phi=0 \text{ on the boundary } C. \tag{3.16}$$

As these equations for ϕ are the same as in Saint-Venant classical solution, except that here $G\theta$ has been replaced by unity, the membrane analogy is still valid. The ordinates of this membrane must however, vary linearly with z. It is easily verified that the external forces $T_x=\tau_{xz}$ and $T_y=\tau_{yz}$ applied at the lower cross section (Fig. 4) have no resultants, namely that

$$\int_A \tau_{xz}\,\mathrm{d}x\mathrm{d}y=\int_A \tau_{yz}\,\mathrm{d}x\mathrm{d}y=0 \tag{3.17}$$

and that they are equivalent to a torsional couple

$$M_z=\int_A(\tau_{yz}x-\tau_{xz}y)\,\mathrm{d}x\mathrm{d}y=-2G\theta\iint_A\phi\,\mathrm{d}x\mathrm{d}y \tag{3.18}$$

We have not yet discussed the volume forces F_x, F_y, that had to be introduced to satisfy eqns (3.10). Replacing in (3.10) τ_{xz} and τ_{yz} by their expressions (3.12), we obtain

$$F_x=-\frac{\partial\tau_{xz}}{\partial z}=-G\theta'\frac{\partial\phi}{\partial y};\ F_y=-\frac{\partial\tau_{yz}}{\partial z}=+G\theta'\frac{\partial\phi}{\partial x}. \tag{3.19}$$

These forces have the same distribution in the cross sections as the shear stresses (3.12). They have therefore no resultants ($R_x=R_y=0$) and their couple around z is equal to the rate of the torsional moment

$$\int_A(F_y\,\mathrm{d}x-F_x\,\mathrm{d}y)\mathrm{d}x\mathrm{d}y=-2G\theta'\int_A\phi\,\mathrm{d}x\mathrm{d}y=M_z'. \tag{3.20}$$

As said in previous section, these volume forces may be replaced by adequate surface tractions producing, per unit length of the axis, the same rate of torsional couple M_z'.

3.3 *Summary*

In summary, above theory shows that:

(1) Saint-Venant's theory is still rigorously applicable to a beam made of the orthotropic material defined in Section 2 if the torsional couple varies linearly along the axis; Membrane analogy is still applicable but, everything else being equal, the cotes z of the membrane vary linearly with z.

(2) The basis difference between the present theory and the classical Saint Venant solution

is that, here, the beam is subjected to axial stresses

$$\sigma_z = E\theta'\psi(x, y) \tag{3.7}$$

which have the same distribution in every cross section. These axial stresses being self-stresses, that is, stresses in equilibrium with no external forces, we have necessarily

$$N \equiv E\theta' \int_A \psi(x, y)\mathrm{d}A = 0;\ M_y = E\theta' \int_A x\psi(x, y)\mathrm{d}A = 0;\ M_x = E\theta' \int_A y\psi(x, y)\mathrm{d}A = 0 \tag{3.21}$$

In fact, these three integral conditions on the warping function $\psi(x, y)$ are only satisfied if the origin 0 coincides with the shear center and the axes Ox, Oy, are parallel to the principal axes of the cross section.

(3) In above theory, no restriction is placed on the nature of the elastic material. The theory is therefore applicable for any kind of orthotropic material, even those for which $k \equiv E/G$ largely exceeds the isotropic value $2(1 + \nu)$.

(4) The only restriction to the validity of the theory is the required existence of infinitely many perfectly rigid cross frames. In particular, the theory applies to thin-walled beams with closed or open cross section. The detailed developments of this possibility are left for another paper.

4. APPLICATIONS OF THE GENERAL THEORY OF TORSION

4.1 *Torsion of a bar with narrow rectangular cross section* (*Fig.* 5)

We assume that $b \gg c$, so that Prandtl's function is represented approximately by a parabolic cylinder whose equation is, by (3.15)

$$\frac{\partial^2 \phi}{\partial x^2} = -2.$$

The integration with $\phi = 0$ for $x = \pm c/2$ gives

$$\phi(x, y) = c^2 - x^2. \tag{4.1}$$

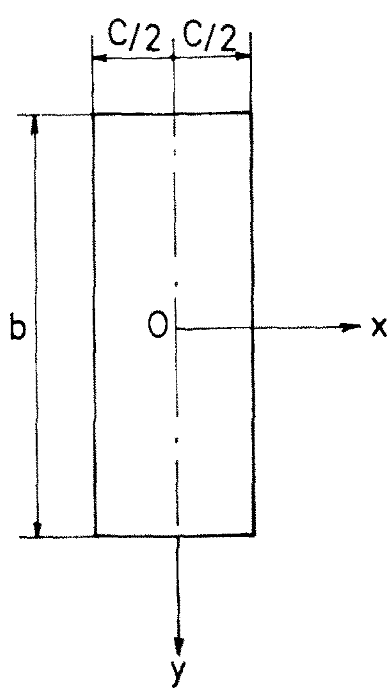

Fig. 5. Narrow rectangular cross section.

By eqns (3.13) and (3.14), we see that the warping function $\psi(x, y)$ satisfies following inequalities

$$\frac{\partial\psi}{\partial x} = y; \frac{\partial\psi}{\partial y} = x. \tag{4.2}$$

The integral of this equation respecting the "self-stressing" conditions (3.21) is

$$\psi(x, y) = xy. \tag{4.3}$$

The warping longitudinal stresses are obtained by (3.7), which reduces here to

$$\sigma_z = E\theta' xy. \tag{4.4}$$

4.2 ***Torsion of a bar with elliptical section*** **(Fig. 6)**

From the classical theory of uniform torsion, we know (see [3], p. 317) that, for an elliptical cross section, Prandtl's stress function is

$$\phi(x, y) = \frac{1}{2}\frac{a^2 - b^2}{a^2 + b^2}(x^2 - y^2) - \frac{1}{2}(x^2 + y^2) \tag{4.5}$$

while the warping function is

$$\psi(x, y) = -\frac{a^2 - b^2}{a^2 + b^2}xy. \tag{4.6}$$

Formula (3.12) then gives

$$\tau_{xz} = -G\theta\frac{2a^2}{a^2 + b^2}y; \ \tau_{yz} = +G\theta\frac{2b^2}{a^2 + b^2}x \tag{4.7}$$

while, by (3.7),

$$\sigma_z = -E\theta'\frac{a^2 - b^2}{a^2 + b^2}xy.$$

The body forces necessary to insure internal equilibrium are, by (3.19):

$$F_x = \frac{2a^2}{a^2 + b^2}G\theta' y \quad F_y = -\frac{2b^2}{a^2 + b^2}G\theta' x. \tag{4.8}$$

If $a = b$, the section of the bar becomes circular. In that case

$$\phi(x, y) = -\frac{1}{2}(x^2 + y^2); \ \psi(x, y) \equiv 0$$

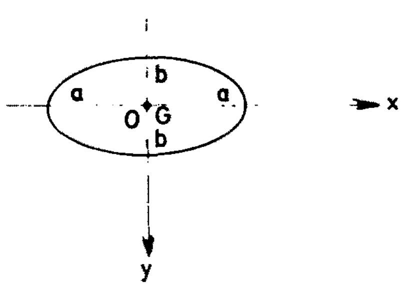

Fig. 6. Elliptical cross section.

and the stresses take their familiar values

$$\tau_{xz} = -G\theta y;\ \tau_{yz} = +G\theta x;\ \sigma_z = 0$$

predicted by Coulomb's theory of torsion.

5. TORSION OF AN ORTHOTROPIC PRISMATIC BAR, TRANSVERSELY RIGID, UNDER THE ACTION OF A TORQUE M_z VARYING ARBITRARILY WITH z

If M_z and θ vary linearly with z, the corresponding axial stresses

$$\sigma_z = E\theta'\psi(x, y) \tag{3.7}$$

do not depend on z. They do not induce corrections $\Delta\tau_{xz}$, $\Delta\tau_{yz}$, nor a warping torsional moment.

On the contrary, as soon as the law of variation of M_z is other than linear, the rate of torsion is non zero and warping stresses are induced. To analyze this situation, let us admit that the displacement field is:

$$u = -\beta y \quad v = +\beta x \quad w = \theta\psi(x, y). \tag{5.1}$$

Incidentally, the assumptions underlying eqns (5.1) are identical to those of the Timoshenko–Vlasov technical theory; indeed, they are:

(1) rotation "en bloc" of each cross section around its shear center;

(2) same warping as in uniform torsion.

The present displacement field induces the strain field

$$\epsilon_x = \epsilon_y = \gamma_{xy} = 0$$

$$\epsilon_z = \theta'\psi(x, y);\ \gamma_{xz} = \theta(z)\left(-y + \frac{\partial\psi}{\partial x}\right);\ \gamma_{yz} = \theta(z)\left(x + \frac{\partial\psi}{\partial y}\right). \tag{5.2}$$

The stress field is therefore:

$$\begin{aligned} \sigma_z &= E\theta'\psi(x, y) \\ \tau_{xz} &= G\theta(z)\left(-y + \frac{\partial\psi}{\partial x}\right) \\ \tau_{yz} &= G\theta(z)\left(x + \frac{\partial\psi}{\partial y}\right). \end{aligned} \tag{5.3}$$

The equilibrium of this stress field requires:

(1) transverse body forces

$$F_x = -G\theta'(z)\frac{\partial\phi}{\partial y};\ F_y = +G\theta'(z)\frac{\partial\phi}{\partial y} \tag{5.4}$$

which are taken again by the transverse frames;

(2) the fulfillment of the third equilibrium equation

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0 \tag{5.5}$$

which, considering expressions (5.3), becomes

$$G\theta(z)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + E\theta''\psi(x, y) = 0. \tag{5.6}$$

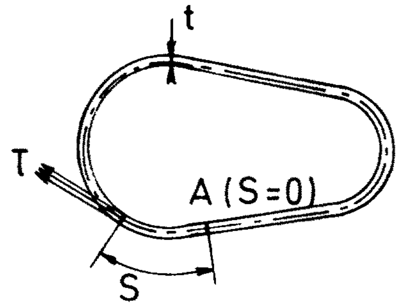

Fig. 7. Thin-walled tubular member subjected to torsion.

In the particular case of a thin-walled beam, the shear stresses are parallel to the median line of the wall and (5.5) may be replaced by the equation (Fig. 7)

$$\frac{\partial \tau}{\partial s} + E\theta''(z)\psi(x, y) = 0. \tag{5.7}$$

Replacing $\theta''(z)$ by $\beta'''(z)$ and integrating from a free edge or a point 0 where $\tau = 0$ (Fig. 7), we obtain:

$$t\tau = -E\beta'''(z)\int_0^s \psi(x, y)\, t\, \mathrm{d}s. \tag{5.8}$$

If we compare this equation with the familiar Timoshenko–Vlasov formula (see [8, 9])

$$t\tau = E\beta'''\int_0^s \omega t\, \mathrm{d}s, \tag{5.9}$$

we see that they are identical provided the warping function $\psi(x, y)$ is replaced by the sectorial coordinate

$$\omega = \omega_s - \bar{\omega}_s \text{ with } \omega_s = \int_0^s r\, \mathrm{d}s \text{ and } \bar{\omega}_s = \frac{1}{A}\int_0^m \omega_s\, t\, \mathrm{d}s \tag{5.10}$$

The physical significance of the sectorial coordinate ω is thus simply to represent warping. Once eqn (5.9) is established, all other equations of the Timoshenko–Vlasov theory follow immediately. In particular, the direct warping stresses are

$$\sigma_z = -E\beta''(z)\omega(s) \tag{5.11}$$

and the warping torsional moment is given by the expression

$$M_z^2 = -E\psi'''\int_0^m \omega^2\, t\, \mathrm{d}s = -C_1\psi''' \text{ with } C_1 = E\int_0^m \omega^2\, t\, \mathrm{d}s \tag{5.12}$$

It must be emphasized that the Timoshenko–Vlasov theory summarized hereabove is approximate, because it neglects the deformational effects of the secondary shear stress (5.9) due to warping restraint. The accuracy of this theory could be measured by evaluating the magnitude of the secondary warping w_2, secondary direct stresses $\Delta\sigma_2$ and secondary shears $\Delta\tau_2$ compared to the magnitude of $w = \omega$, $\sigma = -E\beta''\omega$, $\tau t = E\beta'''\int_0^s \omega t\, \mathrm{d}s$. In the case of open cross sections, the Saint-Venant torsional rigidity C is low, and the warping over the cross section is large. In comparison with the primary, Saint-Venant shear stresses τ_1, the secondary shear stresses τ_2 due to warping restraint and given by (5.9) are small, so that their deformational effects may be neglected. On the other hand, their effects on torsional equilibrium are not negligible, because of the large lever arms with which they act (which are of the order of magnitude of the cross-sectional dimensions).

The torsional behaviour of tubular members is entirely different. The contribution of the shear stresses distributed bi-triangularly over the wall thickness to the internal torsional moment is negligible. The torsional rigidity associated with constant distribution of the τ stresses across the thickness, which was first established by Bredt, is very much greater than the Saint-Venant torsional rigidity of the sliced tube, while the natural warping and the primary torsional stresses are much smaller. Hence, it follows that the warping influences in box sections are much smaller than in open sections and that, nevertheless, the secondary shear stresses (which now, like the primary ones, are uniformly distributed over the wall thickness) are usually of the same order of magnitude as the primary torsional stresses.

In conclusion, in the case of a tubular member, it is not permissible to neglect the deformational influence of the warping shear stresses in relation to that of the primary torsional stresses. Therefore, for a tubular section, formulae (5.1)–(5.8) are rigorous if θ varies linearly and θ' is constant. But, for other laws of distribution of the torsional moment, above formulae differ too much from the rigorous solutions.

In that case, an approximate solution sufficciently accurate for practice has been given by Heilig[4] while von Karman and Chien[5], Argyris and Dunne[6], Benscoter[7] and others have given exact, but very complicated solutions.

6. BENDING WITH SHEAR OF AN ORTHOTROPIC, TRANSVERSELY RIGID, PRISMATIC BAR (FIG. 8)

6.1 *General*

We suppose that the shear force T varies linearly, and therefore the bending moment quadratically, with the longitudinal coordinate z; in agreement with (2.8) and (2.9)

$$T(z) = T_0 - pz \qquad M(z) = -\frac{pz^2}{2} + T_0 z + C. \tag{6.1}$$

The theory will develop about the same general lines as the classical Saint-Venant solution, except that:

(1) The material is orthotropic, transversely rigid, so that the cross sections do not deform in their plane; the stress–strain relations are, as in Section 2:

$$\epsilon_x = \epsilon_y = \gamma_{xy} = 0$$
$$\epsilon_z = \frac{\sigma_z}{E};\ \gamma_{xz} = \frac{\tau_{xz}}{G};\ \gamma_{yz} = \frac{\tau_{yz}}{G}. \tag{6.2}$$

(2) The stresses σ_x, σ_y, τ_{xy} are zero as in Saint-Venant's solution. However, σ_z is not any

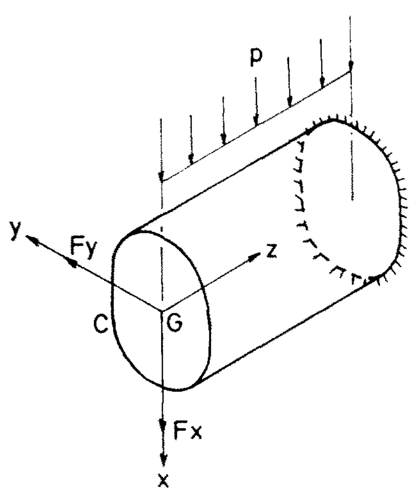

Fig. 8. Bending with shear of a bar.

more given by Navier formula, but contains a corrective term $\Delta\sigma$ representing the distributed shear lag, which does not depend on z. The determination of $\Delta\sigma$ is, incidentally, the most important point of present theory.

In agreement with above assumptions, we postulate that the stress field is

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{6.3}$$

$$\sigma_z = \frac{Mx}{I} + \Delta\sigma(x, y) = \left(-\frac{pz^2}{2} + T_0 z + C\right)\frac{x}{I} + \Delta\sigma(x, y) \tag{6.4}$$

$$\tau_{xz} = \tau_{xz};\ \tau_{yz} = \tau_{yz}. \tag{6.5}$$

It remains to choose the functions $\Delta\sigma$, τ_{xz} and τ_{yz} in such a way that:

(a) the equations of internal equilibrium;
(b) the statical boundary conditions;
(c) the conditions of compatibility

are satisfied.

6.2 *Equations of equilibrium*

Assuming the existence of volume forces F_i having only transverse components, F_x, F_y, we find that the equilibrium equations reduce to:

$$\frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \tag{6.6}$$

$$\frac{\partial \tau_{yz}}{\partial z} + F_y = 0 \tag{6.7}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{x}{I}(-pz + T_0) = 0. \tag{6.8}$$

This shows that it is possible to satisfy these equations by putting

$$\tau_{xz} = (-pz + T_0)\tau_{xo}(x, y);\ \tau_{yz} = (-pz + T_0)\tau_{yo}(x, y). \tag{6.9}$$

Substituting in (6.6) and (6.7), we find volume forces

$$F_x = p\tau_{xo}(x, y);\ F_y = p\tau_{yo}(x, y) \tag{6.10}$$

which are the same in all sections. On the other hand, it is easy to verify that the only non-trivial equation, namely (6.8), is satisfied by following expressions, inspired from Saint-Venant solution

$$\tau_{x0}(x, y) = \frac{\partial \phi}{\partial y} - \frac{x^2}{2I} + f(y);\ \tau_{y0}(x, y) = -\frac{\partial \phi}{\partial x}. \tag{6.11}$$

where $\phi(x, y)$ is a stress function depending on x and y and $f(y)$ is a function of y alone that will be determined later. Volume forces F_x and F_y are absorbed by the rigid transverse frames. It is easy to show that the resultant of these forces is equivalent to the unit vertical transverse force p. Indeed, the further discussion of the boundary conditions (see Section 6.4 hereafter) will show that this unique boundary condition (6.27) reduces to $\phi = 0$ along the contour provided we choose $f(y)$ to annul the right-hand member of (6.27). Then, we have

$$q_y = p\int_A Fy\,\mathrm{d}x\mathrm{d}y = -p\int_A \frac{\partial \phi}{\partial x}\mathrm{d}x\mathrm{d}y = -p\int \mathrm{d}y\int_{\phi_1}^{\phi_2} \frac{\partial \phi}{\partial x}\mathrm{d}x = 0$$

$$q_x = \int_A F_x\,\mathrm{d}x\mathrm{d}y = p\left[\int \mathrm{d}x\int_{\phi_3}^{\phi_4}\frac{\partial \phi}{\partial y}\mathrm{d}y - \frac{1}{2I}\int_A x^2\,\mathrm{d}A + \int_A f(y)\mathrm{d}x\mathrm{d}y\right] =$$

$$p\left[\int_A f(y)\,\mathrm{d}x\mathrm{d}y - \frac{1}{2}\right]. \tag{6.12}$$

We shall show that the last bracket is always equal to unity, so that $q_x = p$ as it should be.

6.3 *Compatibility equations*

The general compatibility equations are ([2])

$$\left\{\begin{aligned} &\frac{\partial^2\epsilon_x}{\partial y^2}+\frac{\partial^2\epsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}\\ &\frac{\partial^2\epsilon_y}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial y^2}=\frac{\partial^2\gamma_{yz}}{\partial y\partial z}\\ &\frac{\partial^2\epsilon_z}{\partial x^2}+\frac{\partial^2\epsilon_x}{\partial z^2}=\frac{\partial^2\gamma_{xz}}{\partial x\partial z}\end{aligned}\right. \qquad (6.13)$$

$$\left\{\begin{aligned} &2\frac{\partial^2\epsilon_x}{\partial y\partial z}=-\frac{\partial^2\gamma_{yz}}{\partial x^2}+\frac{\partial^2\gamma_{xz}}{\partial x\partial y}+\frac{\partial^2\gamma_{xy}}{\partial x\partial z}\\ &2\frac{\partial^2\epsilon_y}{\partial x\partial z}=-\frac{\partial^2\gamma_{xz}}{\partial y^2}+\frac{\partial^2\gamma_{xy}}{\partial y\partial z}+\frac{\partial^2\gamma_{yz}}{\partial x\partial y}\\ &2\frac{\partial^2\epsilon_z}{\partial x\partial y}=-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}+\frac{\partial^2\gamma_{yz}}{\partial x\partial z}+\frac{\partial^2\gamma_{xz}}{\partial y\partial z}\end{aligned}\right. \qquad (6.14)$$

To obtain equations equivalent to the Beltrami–Michell equations, we must replace the ϵ_{ij} by their expressions in terms of the σ_{ij} valid for the orthotropic, transversely rigid material discussed in Section 2. Taking account of eqns (6.3), (6.4), (6.9) and (6.11), we express the strain tensor as follows:

$$\left\{\begin{aligned} &\epsilon_x=0\\ &\epsilon_y=0\\ &\epsilon_z=\frac{x}{EI}\left(-\frac{pz^2}{2}+T_0z+C\right)+\frac{1}{E}\Delta\sigma(x,\ y)\end{aligned}\right. \qquad (6.15)$$

$$\left\{\begin{aligned} &\gamma_{xy}=0\\ &\gamma_{xz}=\frac{1}{G}(-pz+T_0)\left[\frac{\partial\phi}{\partial y}-\frac{x^2}{2I}+f(y)\right]\\ &\gamma_{yz}=-\frac{1}{G}(-pz+T_0)\frac{\partial\phi}{\partial x}\end{aligned}\right. \qquad \left\{\begin{aligned} &\tau_{xy}=0\\ &\tau_{xz}=T\left[\frac{\partial\phi}{\partial y}-\frac{x^2}{2I}+f(y)\right]\\ &\tau_{yz}=-T\frac{\partial\phi}{\partial x}\end{aligned}\right. \qquad (6.16)$$

Replacing in (6.13) and (6.14), we find

$$\left\{\begin{aligned} &0=0\\ &\frac{1}{E}\frac{\partial^2(\Delta\sigma)}{\partial y^2}=\frac{p}{G}\frac{\partial^2\phi}{\partial x\partial y}\\ &\frac{1}{E}\frac{\partial^2(\Delta\sigma)}{\partial x^2}=-\frac{p}{G}\left[\frac{\partial^2\phi}{\partial x\partial y}-\frac{x}{I}\right] \qquad (6.18)\end{aligned}\right.$$

$$\left\{\begin{aligned} &\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^3\phi}{\partial x\partial y^2}=0 \qquad (6.19)\\ &-\frac{\partial^3\phi}{\partial x\partial y^2}-\frac{\partial^3\phi}{\partial y^3}-\frac{\mathrm{d}^2f(y)}{\mathrm{d}y^2}=0 \qquad (6.20)\\ &\frac{2}{E}\frac{\partial^2(\Delta\sigma)}{\partial x\partial y}=\frac{p}{G}\left[\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2}-\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right]. \qquad (6.21)\end{aligned}\right.$$

Equations (6.19) and (6.20) determine $\phi(x,\ y)$, while the three others [(6.17), (6.18), (6.21)] determine $\Delta\sigma(x,\ y)$ and $f(y)$. (6.19) and (6.20) may be written

$$\frac{\partial}{\partial x}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=0 \qquad (6.22)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)=-\frac{\mathrm{d}^2f}{\mathrm{d}y^2}. \qquad (6.23)$$

These relations yield

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=-\frac{\mathrm{d}f}{\mathrm{d}y}+C \tag{6.24}$$

where C is an integration constant. It can be shown [cf [2], p. 309] that, if not torsion occurs, C is equal to zero.

6.4 *Boundary conditions*

It is easily seen that the two first boundary conditions are identically satisfied and that the third gives:

$$\tau_{xz}l+\tau_{yz}m=0. \tag{6.25}$$

Figure 9 shows that

$$l=\cos(\nu,x)=\frac{\mathrm{d}y}{\mathrm{d}s};\ m=\cos(\nu,y)=-\frac{\mathrm{d}x}{\mathrm{d}s},$$

whence (6.25) becomes

$$\tau_{xz}\frac{\mathrm{d}y}{\mathrm{d}s}-\tau_{yz}\frac{\mathrm{d}x}{\mathrm{d}s}=0. \tag{6.26}$$

Introducing in (6.26) the expressions of the stresses resulting from (6.9) and (6.11) and simplifying by $(-pz+T_0)$, we get the equation:

$$\frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s}\left(=\frac{\partial\phi}{\partial s}\right)=\left[\frac{x^2}{2I}-f(y)\right]\frac{\mathrm{d}y}{\mathrm{d}s} \tag{6.27}$$

The values of ϕ along the boundary C are determined by this equation if we choose a definite expression for the function $f(y)$. Differential equation (6.24) plus boundary condition (6.27) determine completely ϕ, as we shall see in Section 7 devoted to practical applications. The simplest way is to choose function $f(y)$ to annul the right-hand member of (6.27).

6.5 *Determination of* $\Delta\sigma(x, y)$

The three remaining compatibility equations read:

$$\frac{\partial^2(\Delta\sigma)}{\partial y^2}=\frac{pE}{G}\frac{\partial^2\phi}{\partial x\partial y} \tag{6.17}$$

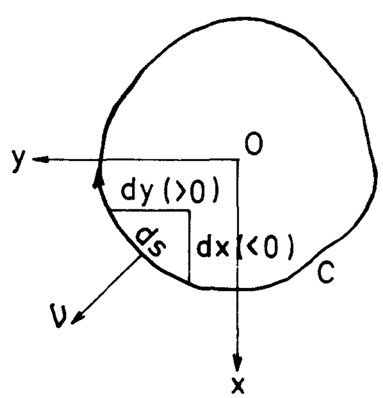

Fig. 9. Boundary conditions.

$$\frac{\partial^2(\Delta\sigma)}{\partial x^2} = -\frac{pE}{G}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{x}{I}\right) \qquad (6.18)$$

$$\frac{\partial^2(\Delta\sigma)}{\partial x\partial y} = \frac{pE}{2G}\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\mathrm{d}f}{\mathrm{d}y}\right). \qquad (6.21)$$

For each type of cross section, one must find the distribution of $\Delta\sigma = \Delta\sigma(x, y)$ which satisfies (6.17), (6.18) and (6.21). This distribution will obviously depend on the value obtained for function $\phi(x, y)$. In Section 7, this technique will be illustrated by studying successively the circular cross section, then the rectangular narrow section.

6.6 *Determination of the displacements-deflection curve of the axis*

The displacement field (u, v, w) is determined, within a rigid movement, by the six first order partial differential equations:

$$\epsilon_x \equiv \frac{\partial u}{\partial x} = 0 \quad (6.28); \; \epsilon_y \equiv \frac{\partial v}{\partial y} = 0 \quad (6.29); \; \gamma_{xy} \equiv \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad (6.30)$$

$$\epsilon_2 \equiv \frac{\partial w}{\partial z} = \frac{\sigma_z}{E} = \frac{x}{EI}\left(-\frac{pz^2}{2} + T_0 z + C\right) + \frac{\Delta\sigma(x, y)}{E} \qquad (6.31)$$

$$\gamma_{xz} \equiv \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G}(-pz + T_0)\left[\frac{\partial\phi}{\partial y} - \frac{x^2}{2I} + f(y)\right] \qquad (6.32)$$

$$\gamma_{yz} \equiv \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\frac{1}{G}(-pz + T_0)\frac{\partial\phi}{\partial x}. \qquad (6.33)$$

Integrating (6.31), we find

$$w = \frac{x}{EI}\left(-\frac{pz^3}{6} + \frac{T_0 z^2}{2} + Cz + D\right) + z\frac{\Delta\sigma(x, y)}{E} + w_0(x, y). \qquad (6.34)$$

Equations (6.32) and (6.33) give then

$$\frac{\partial u}{\partial z} = \frac{1}{G}(-pz + T_0)\left[\frac{\partial\phi}{\partial y} - \frac{x^2}{2I} + f(y)\right] - \frac{1}{EI}\left(-\frac{pz^3}{6} + \frac{T_0 z^2}{2} + Cz + D\right) - \frac{z}{E}\frac{\partial(\Delta\sigma)}{\partial x} - \frac{\partial w_0(x, y)}{\partial x} \qquad (6.35)$$

$$\frac{\partial v}{\partial z} = -\frac{1}{G}(-pz + T_0)\frac{\partial\phi}{\partial x} - \frac{z}{E}\frac{\partial(\Delta\sigma)}{\partial y} - \frac{\partial w_0(x, y)}{\partial y}. \qquad (6.36)$$

Integrating these equations, we find:

$$u = \frac{1}{G}\left(-\frac{pz^2}{2} + T_0 z\right)\left[\frac{\partial\phi}{\partial y} - \frac{x^2}{2I} + f(y)\right] - \frac{1}{EI}\left(-\frac{pz^4}{24} + \frac{T_0 z^3}{6} + \frac{Cz^2}{2} + Dz + F\right) - \frac{z^2}{2E}\frac{\partial(\Delta\sigma)}{\partial x} - z\frac{\partial w_0(x, y)}{\partial x} + f_1(x, y) \qquad (6.37)$$

$$v = -\frac{1}{G}\left(-\frac{pz^2}{2} + T_0 z\right)\frac{\partial\phi}{\partial x} - \frac{z^2}{2E}\frac{\partial(\Delta\sigma)}{\partial y} - z\frac{\partial w_0(x, y)}{\partial y} + f_2(x, y). \qquad (6.38)$$

Introducing these expressions of u and v into the three homogeneous equations (6.28)–(6.30), we obtain 3 lengthy equations that are not reproduced here. The coefficients of z^2, z^1 and z^0 in each of these three equations must separately be equal to zero.

This gives the nine conditions

$$\begin{cases} -\dfrac{p}{2G}\left(\dfrac{\partial^2\phi}{\partial x\partial y}-\dfrac{x}{I}\right)-\dfrac{1}{2E}\dfrac{\partial^2(\Delta\sigma)}{\partial x^2}=0 & (6.39)\\ \dfrac{T_0}{G}\left(\dfrac{\partial^2\phi}{\partial x\partial y}-\dfrac{x}{I}\right)-\dfrac{\partial^2 w_0}{\partial x^2}=0 & (6.40)\\ \dfrac{\partial f_1}{\partial x}=0 & (6.41)\end{cases}$$

$$\begin{cases} \dfrac{p}{2G}\dfrac{\partial^2\phi}{\partial x\partial y}-\dfrac{1}{2E}\dfrac{\partial^2(\Delta\sigma)}{\partial y^2}=0 & (6.42)\\ -\dfrac{T_0}{G}\dfrac{\partial^2\phi}{\partial x\partial y}-\dfrac{\partial^2 w_0}{\partial y^2}=0 & (6.43)\\ \dfrac{\partial f_1(x,\ y)}{\partial y}=0 & (6.44)\end{cases}$$

$$\begin{cases} -\dfrac{p}{2G}\left[\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{\mathrm{d}f(y)}{\mathrm{d}y}\right]-\dfrac{1}{2E}\dfrac{\partial^2(\Delta\sigma)}{\partial x\partial y}+\dfrac{p}{2G}\dfrac{\partial^2 y}{\partial x^2}-\dfrac{1}{2E}\dfrac{\partial^2(\Delta\sigma)}{\partial x\partial y}=0 & (6.45)\\ \dfrac{T_0}{G}\left[\dfrac{\partial^2\phi}{\partial y^2}+\dfrac{df(y)}{\mathrm{d}y}\right]-\dfrac{T_0}{G}\dfrac{\partial^2\varphi}{\partial x^2}-2\dfrac{\partial^2 w_0}{\partial x\partial y}=0 & (6.46)\\ \dfrac{\partial f_1}{\partial y}+\dfrac{\partial f_2}{\partial x}=0 & (6.47)\end{cases}$$

(6.41) gives $f_1 = f_1(y)$, then (6.44) gives $f_1 = K$ (constant); (6.47) gives then $\partial f_2/\partial x = 0$, then $f_2 = f_2(y)$. (6.39) is identically satisfied by (6.18), (6.42) identically satisfied by (6.17) and (6.45) identically satisfied by (6.21). We are thus left with eqns (6.40), (6.43) and (6.46) which should permit the determination of the function $w_0(x,\ y)$. These equations may be written:

$$\frac{\partial^2 w_0}{\partial x^2}=\frac{T_0}{G}\left(\frac{\partial^2\phi}{\partial x\partial y}-\frac{x}{I}\right) \tag{6.48}$$

$$\frac{\partial^2 w_0}{\partial y^2}=-\frac{T_0}{G}\frac{\partial^2\phi}{\partial x\partial y} \tag{6.49}$$

$$\frac{\partial^2 w_0}{\partial x\partial y}=\frac{T_0}{2G}\left[\frac{\partial^2\phi}{\partial y^2}-\frac{\partial^2\phi}{\partial x^2}+\frac{\mathrm{d}f(y)}{\mathrm{d}y}\right]. \tag{6.50}$$

This shows that $w_0(x,\ y)$ is at any point proportional to $\Delta\sigma(x,\ y)$. Indeed, eqns (6.48)–(6.50) become identical to eqns (6.17), (6.18) and (6.21) if one puts

$$w_0(x,\ y)=-\frac{T_0}{pE}\Delta\sigma(x,\ y)+\text{arbitrary linear function.} \tag{6.51}$$

If we adopt this expression for $w_0(x,\ y)$, we see that the displacement field is completely defined by eqns (6.34), (6.37) and (6.38), that are rewritten as follows:

$$u=\frac{1}{G}\left(-\frac{pz^2}{2}+T_0 z\right)\left[\frac{\partial\phi}{\partial y}-\frac{x^2}{2I}+f(y)\right]-\frac{1}{EI}\left(-\frac{pz^4}{24}+\frac{T_{0z^3}}{6}+\frac{Cz^2}{2}+Dz+F\right)-\left(\frac{z^2}{2E}-\frac{zT_0}{pE}\right)\frac{\partial(\Delta\sigma)}{\partial x}+K \tag{6.52}$$

$$v=-\frac{1}{G}\left(-\frac{pz^2}{2}+T_0 z\right)\frac{\partial\phi}{\partial x}-\left(\frac{z^2}{2E}-\frac{zT_0}{pE}\right)\frac{\mathrm{d}(\Delta\sigma)}{\mathrm{d}y} \tag{6.53}$$

$$w=\frac{x}{EI}\left(-\frac{pz^3}{6}+T_0\frac{z^2}{2}+Cz+D\right)+\left(\frac{z}{E}-\frac{T_0}{pE}\right)\Delta\sigma(x,\ y). \tag{6.54}$$

The deflection curve of the beam is $U(z) \equiv [u]_{y=0}^{x=0}$. Equation (6.52) gives

$$U(z) = \frac{1}{EI}\left(\frac{pz^4}{24} - \frac{T_0 z^3}{6} - \frac{Cz^2}{2}\right) + K + \frac{1}{G}\left(-\frac{pz^2}{2} + T_0 z\right)\left\{\left(\frac{\partial \phi}{\partial y}\right)_{\substack{x=0\\y=0}} + f(y)_{y=0}\right\}$$

$$-\left(\frac{z^2}{2E} - \frac{zT_0}{pE}\right)\left[\frac{\partial(\Delta\sigma)}{\partial x}\right]_{\substack{x=0\\y=0}} \tag{6.55}$$

Taking account of expression (6.1) of the bending moment, one verifies easily that the first term of the right hand member of (6.55) is the classical deflection, obtained by integrating the well known differential equation

$$\frac{d^2 U}{dz^2} = -\frac{M}{EI}. \tag{6.56}$$

The second term of this right hand member should therefore be the correction ΔU due to the shear deformation.

In the elementary theory proposed by Timoshenko, ΔU obeys the differential equation

$$\frac{d^2(\Delta U)}{dz^2} = -\frac{p}{GA'}, \tag{6.57}$$

where A' is the so-called reduced section for shear.

The integral of (6.57) is

$$\Delta U = -\frac{pz^2}{2GA'} + Qz + R \tag{6.58}$$

where Q and R are integration constants. Comparing (6.58) with the second term of (6.55), we see that we must have

$$\frac{1}{A'} = \left(\frac{\partial \phi}{\partial y}\right)_{\substack{x=0\\y=0}} + f(y)_{y=0} + \frac{G}{pE}\left[\frac{\partial(\Delta\sigma)}{\partial x}\right]_{\substack{x=0\\y=0}}. \tag{6.59}$$

In the common case of beams with doubly symmetrical sections, $\partial(\Delta\sigma)/\partial x$ is an even function in x, which is zero for $x = 0$. In this particular case, one finds that the exact value of the reduced section for shear is

$$A' = \frac{1}{[f(y)]_{y=0} + \left(\frac{\partial \phi}{\partial y}\right)_{\substack{x=0\\y=0}}}. \tag{6.60}$$

7. PRACTICAL APPLICATIONS

7.1 *Bending with shear of a beam with circular cross section*

7.1.1 *General.* Let

$$x^2 + y^2 = r^2 \tag{7.1}$$

be the equation of the circular contour of the cross section. The right hand member of boundary condition (6.27) becomes zero if we take

$$f(y) = \frac{1}{2I}(r^2 - y^2). \tag{7.2}$$

Replacing in eqn (6.24), one sees that the stress function is defined by equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{y}{I} \tag{7.3}$$

and the condition $\phi = 0$ on the boundary. It is easy to see that eqn (7.3) is satisfied by taking for stress function the expression

$$\phi = m(x^2 + y^2 - r^2)y \tag{7.4}$$

where m is a constant. This function is zero on the boundary (7.1) and satisfies eqn (7.3) if we take

$$\phi(x, y) = \frac{1}{8I}(x^2 + y^2 - r^2)y. \tag{7.5}$$

7.1.2. *Computation of the shearing stresses.* The shear stresses τ_{xz} and τ_{yz} are deduced from (7.3) by eqns (6.9) and (6.11). We obtain

$$\tau_{xz} = (-pz + T_0)\frac{1}{8I}(3r^2 - 4x^2 - y^2)$$

$$\tau_{yz} = -(pz + T_0)\frac{xy}{4I}. \tag{7.6}$$

The vertical shearing stress τ_{xz} is an even function of x and y and the horizontal shearing stress, τ_{yz}, an odd function of the same variables.

Along the horizontal diameter of the cross section, $x = 0$ and one finds by eqns (7.6)

$$(\tau_{xz})_{x=0} = \frac{-pz + T_0}{8I}(3r^2 - y^2). \tag{7.7}$$

The maximum shearing stress occurs at the center ($y = 0$), where

$$(\tau_{xz})_{\max} = \frac{3Tr^2}{8I} = 0.375\frac{Tr^2}{I} = 1.5\frac{T}{A}. \tag{7.8}$$

It is interesting to compare this value with the value obtained by Saint-Venant for an isotropic material, namely ([2])

$$(\tau_{xz})_{\max}^{\text{isotropic}} = \frac{3 + 2\nu}{8(1 + \nu)}\frac{Tr^2}{I} = 0.346\frac{Tr^2}{I} \text{ (for } \nu = 0.3) \tag{7.9}$$

The shearing stress at the ends of the horizontal diameter ($y = \pm r$) is

$$(\tau_{xz})_{y=\pm r} = \frac{Tr^2}{4I} = 0.250\frac{Tr^2}{I} = \frac{T}{A}. \tag{7.10}$$

The isotropic value found by Saint-Venant is

$$(\tau_{xz})_{y=\pm r}^{\text{isotropic}} = \frac{1 + 2\nu}{4(1 + \nu)}\frac{Tr^2}{I} = 0.308\frac{Tr^2}{I} \text{ (for } \nu = 0.3) \tag{7.11}$$

The elementary theory given in Mechanics of Materials leads to the constant value

$$(\tau_{xz})_{x=0} = \frac{Tr^2}{3I} = 0.333\frac{Tr^2}{I} = \frac{4}{3}\frac{T}{A} \tag{7.12}$$

along the horizontal diameter.

One can easily verify that

$$\int_A \tau_{xz}\,\mathrm{d}A = T \tag{7.13}$$

which means that the τ_{xz} stresses equilibrate exactly the shear force. It can also be verified that the volume forces F_x given by (6.10) are equivalent to a constant transverse force p. Indeed, introducing the expression (7.2) of $f(y)$ into eqn (6.12), we obtain

$$q_x = p\left[\int_A \frac{1}{2I}(r^2 - y^2)\,\mathrm{d}A - \frac{1}{2}\right] = p$$

because $A = \pi r^2$ and $I = \pi r^4/4$.

7.1.3 *Computation of the corrective stress $\Delta\sigma$ representing the shear lag.* The corrective stress $\Delta\sigma(x, y)$ must satisfy conditions (6.17), (6.18) and (6.21). Taking account of the expressions (7.5) of $\phi(x, y)$ and (7.2) of $f(y)$, we get

$$\left\{\begin{aligned} &\frac{\partial^2(\Delta\sigma)}{ay^2} = \frac{pE}{4GI}x \\ &\frac{\partial^2(\Delta\sigma)}{\partial x^2} = \frac{3pE}{4GI}x \\ &\frac{\partial^2(\Delta\sigma)}{\partial x \partial y} = \frac{pEy}{4GI}. \end{aligned}\right. \tag{7.14}$$

Integrating these equations, we find respectively:

$$\left\{\begin{aligned} &\Delta\sigma = \frac{pExy^2}{8GI} + yf(x) + g(x) \\ &\Delta\sigma = \frac{pEx^3}{8GI} + xh(y) + k(y) \\ &\Delta\sigma = \frac{pExy^2}{4GI} + 1(x) + m(y), \end{aligned}\right. \tag{7.15}$$

where f, g, h, k, l and m are arbitrary functions. The three expressions above are satisfied if we take

$$\Delta\sigma(x, y) = \frac{pE}{8GI}x(x^2 - y^2). \tag{7.16}$$

This expression satisfies following necessary conditions:
be proportional to p;
be proportional to E/G;
be odd in x.

7.1.4 *Value of the reduced section for shear.* Applying eqn (6.60) and using expression (7.1) of $f(y)$, and expressions (7.5) of $\phi(x, y)$, we find

$$\left(\frac{\partial\phi}{\partial y}\right)_{\substack{x=0\\y=0}} = -\frac{r^2}{8I};\ [f(y)]_{y=0} = \frac{r^2}{2I} \text{ and, therefore } A' = \frac{2\pi r^2}{3} = \frac{2}{3}A. \tag{7.17}$$

7.2 *Bending with shear of a beam with narrow rectangular section* (Fig. 10)

We treat essentially this case for control, because we must retrieve the solution obtained directly by considering the problem as a problem of plane stress (see companion paper

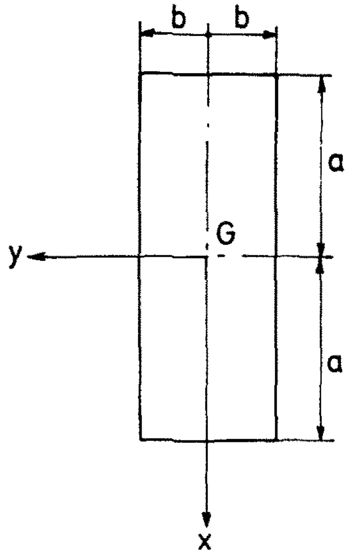

Fig. 10. Rectangular cross section.

The boundary of the rectangle has for equation

$$(x^2-a^2)(y^2-b^2)=0. \tag{7.18}$$

If, in eqn (6.27) we replace $f(y)$ by the constant $a^2/2I$, the expression

$$\frac{x^2}{2I}-\frac{a^2}{2I}$$

becomes zero along the sides $x=\pm a$ of the rectangle. Along the vertical sides $y=\pm b$, the derivative $\mathrm{d}y/\mathrm{d}s$ is zero. The right hand member of boundary condition (6.27) is thus zero and we may take $\phi=0$ on the boundary. The differential equation (6.24) reduces then to

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0. \tag{7.19}$$

When a is large with respect to b (Fig. 10), we can accept that, for points sufficiently remote from the small sides of the rectangle, the membrane representing ϕ is substantially cylindrical. Equation (7.19) reduces then to

$$\frac{\partial^2\phi}{\partial y^2}=0$$

and its general integral $\phi=Ay+B$ is zero because ϕ must become nought for $y=\pm b$. The first formula (6.16) gives then simply

$$\tau_{xz}=\frac{T}{2I}(a^2-x^2)=\frac{3T}{2A}\left[1-\left(\frac{x}{a}\right)^2\right] \tag{7.20}$$

that is the parabolic distribution of Mechanics of Materials. We have now to determine the correction $\Delta\sigma$, identical to the distributed shear lag. Introducing $\phi=0$ and $f(y)=a^2/2I$ into eqns (6.17), (6.18) and (6.21), we find:

$$\frac{\partial^2(\Delta\sigma)}{\partial y^2} = 0$$
$$\frac{\partial^2(\Delta\sigma)}{\partial x^2} = \frac{pE}{GI}x \qquad (7.21)$$
$$\frac{\partial^2(\Delta\sigma)}{\partial x \partial y} = 0.$$

The most general expression satisfying the three eqns (7.21) is

$$\Delta\sigma = \frac{pE}{6IG}x^3 + Mx + N. \qquad (7.22)$$

The arbitrary constants M and N are determined by stipulating that the $\Delta\sigma$ must have ze resultant and zero moment. This gives

$$N = 0,\ M = \frac{a^2}{10}\frac{E}{I}.$$

Therefore (7.22) becomes

$$\Delta\sigma = \frac{pE}{4IG}\left(\frac{2}{3}x^3 - \frac{2}{5}a^2x\right) \qquad (7.23)$$

what is exactly the expression found in the companion paper [10].

REFERENCES

1. S. G. Lekhnitskii, *Theory of Elasticity of an Anisotropic Elastic Body.* Holden-Day, San Francisco (1963).
2. S. P. Timoshenko, *Theory of Elasticity. Théorie de l'Elasticité* (Edited by Béranger). Paris et Liège (1936).
3. A. E. H. Love, *The Mathematical Theory of Elasticity,* 4th Edn. Dover, New York (1944).
4. R. Heilig, Beitrag zur Theorie der Kastenträger beliebiger Querschnittsform. *Der Stahlbau* **30**, 333–349 (1961).
5. Th. Von Karman and W. Z. Chien, Torsion with variable Twist. *J. Aero. Sci.* **13**, 503–510 (1946).
6. J. H. Argyris and P. C. Dunne, The general theory of cylindrical and conical tubes under torsion and bending loads. *J. Roy. Aero. Soc.* **51**, 757–784 (1947).
7. S. U. Benscoter, A theory of torsion bending for multicell beams. *J. Appl. Mech.* **21**, 25–34 (1954).
8. V. Z. Vlasov, *Thin-walled Elastic Beams.* Israël Program for Scientific Translation, Jerusalem (1961).
9. C. E. Massonnet, *Résistance des Matériaux,* Vol. 2, 2nd Edn. Sciences et Lettres, Liège (1973).
10. C. E. Massonnet, Théorie perfectionnée des poutres droites élastiques à parois minces (to appear in IADSE Publications, end of 1982).